C3-R4: MATHEMATICAL METHODS FOR COMPUTING

NOTE:

1.	**Answer question 1 and any FOUR from questions 2 to 7.**
2.	**Parts of the same question should be answered together and in the same sequence.**

Time: 3 Hours **Total Marks: 100**

1.

a) A fair coin is tossed four times. Find the probability that they are all heads if the first two tosses results in head.

b) Consider two independent integer-valued random variables, X and Y. Variable X takes on only the values of the eight integers {1, 2, ..., 8} and does so with uniform probability. Variable Y may take the value of any positive integer k, with probabilities P{Y = k} = 2^{-k}, k = 1, 2, 3, ... Calculate (i) entropies H(X) and H(Y), (ii) the joint entropy H(X, Y) and (iii) their mutual information I(X; Y).

c) The probability distribution of a random variable *X* is given by $f(x) = 2kx^3,\ 0 \le x \le 1$. Find (i) the value of *k* ((ii) $P(X < 1/2)$, (iii) expected value of *X.*

d) The autocorrelation function of a stationary process {*X*(*t*)} is given by $R(\tau) = 9 + 16e^{-2|\tau/3|}$, find mean and standard deviation of *X*(*t*).

e) A factory manufactures two types of cylinders, C1 and C2.Three materials M1, M2 and M3 are required for manufacture of each cylinder. The quantities of materials required are as follows:

	M1	M2	M3
C1	1	1	2
C2	5	2	2

The available units of the quantities M1 M2 M3 are respectively 45, 21 and 24. If the profits earned on each unit of C1 and C2 are respectively Rs 50 and Rs 40, formulate the problem as an LP model to maximize the profit.

f) Find the Fourier transform of the function $f(t) = e^{-2t^2}$.

g) If the customers arrive at a bank counter according to Poisson process with mean rate of 2 per minute, find the probability that the interval between two successive arrivals is (i) at least 1 minute and (ii) at most 4 minutes.

(7x4)

2.

a) A company produces mobiles in its three plants A, B and C with daily production volume of 500; 1000; and 2000 units respectively. According to past experience, it is known that the fractions of defective output produced by these plants are respectively 0.005; 0.008 and 0.010. If a mobile is selected from day's total production and found to be defective, find out (i) from which plant does the mobile come? (ii) what is the probability that it came from plant B?

b) A source without memory has six characters with following probabilities of transmission:

A	B	C	D	E	F
1/3	¼	1/8	1/8	1/12	1/12

Devise the Shannon-Fano encoding procedure to obtain uniquely decodable code to the above message ensemble. Obtain the average length, entropy and efficiency of the code that you obtain.

(9+9)

3.

a) Use simplex method to solve the following linear programming problem:

Maximize $Z = 4x_1 + 10x_2$,

Subject to

$$2x_1 + x_2 \leq 50,$$
$$2x_1 + 5x_2 \leq 100,$$
$$2x_1 + 3x_2 \leq 90,$$
$$x_1, \; x_2 \geq 0.$$

b) Find a Fourier series for the following function:

$$f(x) = \begin{cases} -\pi, & if \; -\pi < x < 0 \\ x, & if \; 0 < x < \pi. \end{cases}$$

(9+9)

4.

a) There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are interchanged. The state of related Markov chain is the number of red marbles in urn A, after the interchange. Find the transition probability matrix of the system. What is the probability that there are 2 red marbles in urn A after 3 steps? Also find the probability that there are 2 red marbles in urn A after long run.

b) In a network of three service stations A, B, C customers arrive at A, B, C from outside in accordance with the Poisson process having rates 5, 10, 15 respectively. The service times at the three stations are exponential with respective rates 10, 50, 100. A customer completing service at station A is equally likely to (i) go to station B, (ii) go to station C, or (iii) leave the system. A customer departing from the service at station B always goes to station C. A customer departure from service at station C is equally likely to go to station B or leave the system. Find the average number of customers in the system consisting of all the 3 stations and the average time a customer spends in the system.

(9+9)

5.

a) The joint PDF of a two dimensional random variable (X, Y) is given by:

$$f(x,y) = \begin{cases} xy^2 + \frac{1}{8}x^2, & 0 < x < 2, \; 0 < y < 1 \\ 0, & elsewhere \end{cases}$$

Find (i) $P(Y < \frac{1}{2} / X > 1)$, (ii) $P(X + Y < 1)$ and (iii) $P(Y > X)$.

b) On an average 96 patient per 24-hour day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes for active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from 4/3 patients to 1/2 patient.

(9+9)

6.

a) If $U(t) = X\cos t + Y\sin t$ and $V(t) = Y\cos t + X\sin t$, where X and Y are independent random variables such that $E(X) = 0, E(Y) = 0, E(X^2) = E(Y^2) = 1$. Show that $\{U(t)\}$ and $\{V(t)\}$ are individually stationary in the wide sense, but not jointly stationary.

b) Use dual simplex method to solve the following linear programming problem:

Minimize $Z = 3x_1 + x_2$,

subject to

$x_1 + x_2 \geq 1$,

$2x_1 + 3x_2 \geq 2$

$x_1,\ x_2 \geq 0$.

(9+9)

7.

a) Use Laplace transform to solve the following initial value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - y = \sin t, \text{ given that } y = 0,\ \frac{dy}{dt} = 0 \text{ at } t = 0.$$

b) Use the KKT conditions to solve the following nonlinear programming problem:

Minimize $Z = x_1^2 + 2x_2^2 + 3x_3^2$

subject to

$x_1 + x_2 \geq 6$,

$x_1 \geq 2,\ x_2 \geq 1$.

(9+9)